The Broadcast Dimension of Graphs

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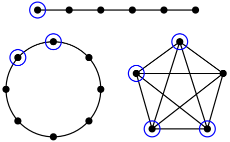

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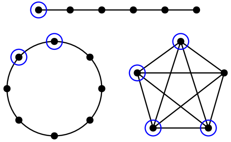

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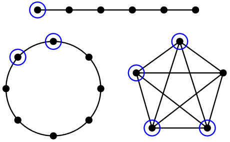

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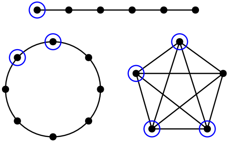

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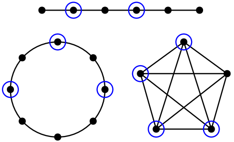

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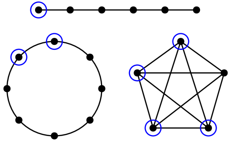

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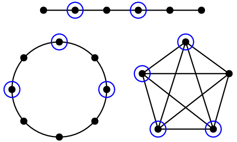

Definition (Geneson-Yi, 2020)

Function $f : V(G) \to \mathbb{Z}^+ \cup \{0\}$ is a **resolving broadcast** of G if, for any distinct $x, y \in V(G)$, there exists a vertex z such that $f(z) > 0$ and $d_{f(z)}(x,z) \neq d_{f(z)}(y,z)$. The **broadcast dimension** $\mathrm{bdim}(G)$ of G is the minimum of $\sum_{v \in V(G)} f(v)$ over all resolving broadcasts f of G.

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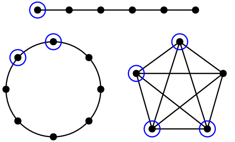

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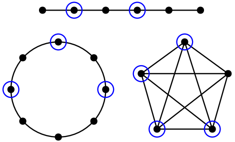

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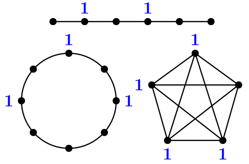

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   The minimum total cost of transmitters required for the robot to determine its location is $\operatorname{bdim}(G)$.

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Theorem (Z)

For all acyclic graphs F of order n, we have

$$\mathrm{bdim}(F) = \Omega(\sqrt{n}),$$

and this lower bound is asymptotically optimal.

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Theorem (Geneson-Yi, 2020)

For the the d-dimensional grid graph $G_k = \Pi_{i=1}^{d} P_k$, we have $\text{bdim}(G_k) = \Theta(k)$ and $\text{adim}(G_k) = \Theta(k^d)$ for every $k \in \mathbb{Z}^+$ and any $d \geq 1$, where the constants in the bounds depend on d.

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Corollary (Z)

There does not exist a family of acyclic graphs $\{G_k\}_{k\in\mathbb{Z}^+}$ with $\text{bdim}(G_k) = k$ and $\text{adim}(G_k) = 2^{\Omega(k)}$ for every $k \in \mathbb{Z}^+$.

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Recall that $\text{bdim}(G) = \Omega(\log n)$ for all graphs G of order n. Thus, my construction has broadcast dimension that is asymptotically optimal in both its order and its adjacency dimension.

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Is $\frac{\text{bdim}(G-v)}{\text{bdim}(G)}$ bounded from above for all graphs G and any vertex $v \in V(G)$?

Thank You!

This research was conducted at the 2020 University of Minnesota Duluth REU program. I extend my thanks to Joe Gallian for organizing the program and for suggesting the problem, as well as the advisors, Amanda Burcroff, Colin Defant, and Yelena Mandelshtam, for their mentorship.

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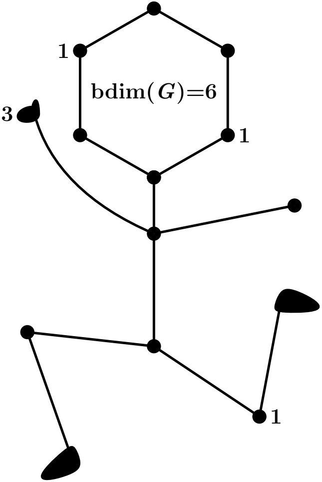

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