Maximizing the Product of the Elements of a Constrained Multiset

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Abstract

We examine multisets of n numbers $x_1, x_2, \ldots, x_n$ in the interval $[0, 1]$ that maximize the product of the elements in the multiset, with the property that for any α, $\sum_{\alpha \leq x_i < 2\alpha} x_i \leq 1$. Because the answer depends strongly on n, we begin by investigating what happens for specific values of n. We then provide upper and lower bounds for the value of the product for larger n. Additionally, we consider two modified versions of this problem: a restricted version where all elements must be unit fractions and a relaxed version where non-integral copies of elements are allowed in the multisets.

1 Introduction

In this paper, we investigate the maximal product of sets of numbers in the interval $[0, 1]$ where the sum of the numbers within various subintervals is constrained by an upper bound. The effect of such a constraint is that the elements must be spread throughout the interval $[0, 1]$, while the maximization of the product compels the numbers to be close to 1. More formally, the maximization of the objective can be achieved by studying the multiset (a set that allows for multiple instances of its elements) $S(n)$ defined as follows.

Definition 1.1. *The multiset $S(n)$ for any positive integer n is a multiset of n numbers $x_1, x_2, \ldots, x_n \in [0, 1]$ that maximizes*

$$\prod_{i=1}^{n} x_i$$

with the property that for any α,

$$\sum_{\alpha \leq x_i < 2\alpha} x_i \leq 1.$$

Correspondingly, $P(n)$ is the product of the elements of $S(n)$.

One way we might attempt to maximize the product is by selecting elements *greedily*, that is, selecting the largest element possible on each step until we have n elements. For example, for $n = 10$, we choose the first element to be 1. Then, we cannot include any other elements in the interval $(\frac{1}{2}, 1]$ because any such element $\frac{1}{2} < e_1 \leq 1$ would be in a shared interval $[e_1, 2e_1)$ with the element

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1, and $e_1 + 1$ is clearly greater than 1. The next largest element we can add is $\frac{1}{2}$, and we can add two copies of $\frac{1}{2}$ without breaking the constraint. Similar to before, we cannot include any other elements in the interval $(\frac{1}{4}, \frac{1}{2}]$ because any such element $\frac{1}{4} < e_2 \leq \frac{1}{2}$ would be in a shared interval of $[e_2, 2e_2)$ with the two $\frac{1}{2}$'s, and the sum in that interval would exceed 1. We continue by adding four $\frac{1}{4}$'s and finish up the ten elements with three $\frac{1}{8}$'s.

This multiset, $\{1, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}\}$ is an example of a *feasible* multiset for $n = 10$, where a feasible multiset is a multiset that satisfies the sum constraint from Definition 1.1.

We notice that a greedy feasible multiset consists of 2^i copies of the element $\frac{1}{2^i}$ for every distinct element $\frac{1}{2^i}$ in the multiset, except for the smallest distinct element. The smallest distinct element only follows this pattern when $n = 2^k - 1$ for positive integer values of k. In Section 2, we explore $S(2^k - 1)$ for small values of k. We further conjecture a closed form description of $P(2^k - 1)$ and a corresponding $S(2^k - 1)$.

With further investigation of values of general n that are not necessarily of the form $2^k - 1$, we discover that for some n we can create feasible multisets whose elements have a larger product than the elements of the greedy multisets. For $n = 70$, we see that one such improved feasible solution contains one copy of 1, two copies of $\frac{1}{2}$, four copies of $\frac{1}{4}$, nine copies of $\frac{1}{9}$, eighteen copies of $\frac{1}{18}$, and thirty-six copies of $\frac{1}{36}$. This shows that for general n, the greedy feasible multiset is not always optimal.

In Section 3, we continue to explore $S(n)$ for general n and provide upper and lower bounds on $P(n)$. We also analyze the asymptotic behavior of the negative log of the function $P(n)$, taking the negative log because $P(n)$ is shrinking rapidly in n. We find that the negative log of $P(n)$ grows as $n \log n$.

In Section 4, we wish to computationally generate $S(n)$ for more values of n, so we attempt to formulate the original problem as a linear program. However, the original problem contains an infinite number of constraints. In order to translate the problem into a linear program, we add the additional constraint that all elements of $S(n)$ must be unit fractions, i.e. fractions taking the form $\frac{1}{z}$ where z is an integer. This new restricted problem is formalized as follows.

Definition 1.2 (Restricted Problem)**.** *The multiset $S'(n)$ for any positive integer n is a multiset of n **unit fractions** $x_1, x_2, \ldots, x_n \in [0, 1]$ that maximizes*

$$\prod_{i=1}^{n} x_i$$

with the property that for any α,

$$\sum_{\alpha \leq x_i < 2\alpha} x_i \leq 1.$$

Correspondingly, $P'(n)$ is the products of the elements of $S'(n)$.

In our initial explorations of computational results, it appeared that the sum of the occurrences of each distinct element (except possibly the smallest element) in $S'(n)$ is 1, and that the value of the

$(k+1)$th largest distinct element was at most half the value of the kth largest distinct element. The smallest value of n that disproves either notion is $n = 138$. $S'(138)$ contains one copy of 1, two copies of $\frac{1}{2}$, four copies of $\frac{1}{4}$, eight copies of $\frac{1}{8}$, eight copies of $\frac{1}{16}$, nine copies of $\frac{1}{18}$, sixteen copies of $\frac{1}{32}$, eighteen copies of $\frac{1}{36}$, thirty-two copies of $\frac{1}{64}$, thirty-six copies of $\frac{1}{72}$, and four copies of $\frac{1}{128}$. In $S'(138)$ we see that the sum of the elements equal to $\frac{1}{16}$ is $\frac{1}{2}$, and that $\frac{1}{18} > \frac{1}{2} \times \frac{1}{16}$. Computing $S'(n)$ for larger values of n reveals that in general, we find that $S'(n)$ is often different from the greedy feasible multiset when n is far from a power of 2.

Additionally, we take the dual of the primal linear program of the original problem and construct a feasible solution to the dual for $n = 2^k - 1$ for any positive integer k. By Weak Duality, the objective that the feasible solution to the dual achieves is an upper bound on $P'(n)$.

In Section 5, we examine another modified version of the problem. Here, we relax the constraint requiring the number of copies of each element in a multiset to be an integer. Instead, we now allow fractional copies of elements, and as a result, there is an unbounded number of distinct elements. This new relaxed problem is formalized as follows.

Definition 1.3 (Relaxed Problem)**.** *The multiset $S''(n)$ for any positive integer n is a multiset of n_i copies of x_i for integer $i \geq 1$ that maximizes*

$$\prod_{i=1}^{\infty} (x_i)^{n_i}$$

with the property that for any α,

$$\sum_{\alpha \leq x_i < 2\alpha} n_i x_i \leq 1,$$

and subject to the constraint that

$$\sum_{i=1}^{\infty} n_i = n$$

Correspondingly, $P''(n)$ is the product of the elements of $S''(n)$.

For this relaxed problem, we show that $S''(n)$ contains at least one element in all intervals of the form $[\alpha, 2\alpha)$ for all $\frac{\min S''(n)}{2} < \alpha \leq \max S''(n)$, and we conjecture about the general form of the solution to this relaxed problem.

2 Multisets $S(n)$ where $n = 2^k - 1$

In the introduction, we constructed feasible multisets by generating elements greedily. We notice that each element in these greedy multisets takes the form $\frac{1}{2^i}$. The greedy multiset contains 2^i copies of each element $\frac{1}{2^i}$, except for the smallest element. When constructing the greedy multiset for $n = 2^k - 1$ for positive integer k, the smallest element also follows this pattern: the greedy multiset contains 2^{k-1} copies of the smallest element $\frac{1}{2^{k-1}}$ because $n = 2^k - 1 = 2^0 + 2^1 + \cdots + 2^{k-1}$. We formalize the greedily constructed multiset for $n = 2^k - 1$ as follows:

Definition 2.1. *The multiset G_k consists of 2^i copies of $\frac{1}{2^i}$, for all integers $0 \leq i < k$.*

In this section, we examine $S(2^k-1)$ for small integral values of k, and we then conjecture more generally about this case. We begin with some lemmas that will reduce later casework. First, we prove that 1 is always an element of $S(n)$.

Lemma 2.2. *For all positive integers n, $S(n)$ contains the number 1.*

Proof. For $n = 1$, the product is maximized when the multiset consists of the largest possible element, 1. In the remainder of the proof, we consider $n > 1$.

Assume, for the sake of contradiction, that there exists a positive integer m such that $c = \max(S(m)) < 1$. We consider two cases for the value of c.

Case 1: If $c \leq \frac{1}{2}$, we can replace any element in $S(m)$ with 1 without violating the constraint and increase the product of the multiset. Then, there exists a feasible multiset with a product larger than $P(m)$, so we have reached a contradiction.

Case 2: If $\frac{1}{2} < c < 1$, the second largest element in $S(m)$ is at most $\max\{\frac{c}{2}, 1-c\} < \frac{1}{2}$. Because all but the largest element in $S(m)$ are less than $\frac{1}{2}$, we see that we can replace the largest element c with 1 without violating the constraint. Then, we have constructed a feasible multiset with a product larger than $P(m)$, so we have reached a contradiction. □

Now, we examine some feasible multisets for $n = 2^k - 1$ that never have a greater product than that of G_k. In order to examine the products of feasible multisets F, we define $Q(F)$ to be the product of the elements of F.

Lemma 2.3. *For any feasible multiset F_k with 2^k-1 elements, such that the number of elements in F_k in the interval $\left(\frac{1}{2^i}, 1\right]$ is less than or equal to 2^i-1 for all $0 < i < k$, we have that $Q(G_k) \geq Q(F_k)$.*

Proof. Given a distinct element $\frac{1}{2^i}$ in G_k, we know that there are 2^i copies of that element. Further, we know that there are $2^i - 1$ elements larger than $\frac{1}{2^i}$ in G_k. So, the (2^i)th to $\left(2^{i+1}-1\right)$th largest elements in G_k are equal to $\frac{1}{2^i}$.

From the condition on F_k, we know that the $\left(2^i\right)$th largest element of F_k is at most 2^i. So, the $\left(2^i\right)$th to the $\left(2^{i+1}-1\right)$th largest elements of F_k are less than or equal to the corresponding elements of G_k, for all integers $0 < i < k$. Thus, when F_k and G_k are sorted, every element in F_k is less than or equal to the corresponding element in G_k, so $Q(G_k) \geq Q(F_k)$. □

We also note that since G_k is a feasible multiset for $n = 2^k - 1$, we can use the product of its elements to lower bound $P\left(2^k-1\right)$.

Lemma 2.4. *We have that $P\left(2^k-1\right) \geq 2^{(2-k)2^k-2}$ for positive integer k.*

Proof. The product $P\left(2^k-1\right)$ must be at least as large as $Q(G_k)$, so

$$P\left(2^k-1\right) \geq Q(G_k) = \prod_{i=0}^{k-1} \left(\frac{1}{2^i}\right)^{2^i} = \frac{1}{2^{\sum_{i=0}^{k-1} i2^i}} = \frac{1}{2^{(k-2)2^k+2}} = 2^{(2-k)2^k-2}$$

using the fact that $\sum_{i=0}^{k-1} i2^i = (k-2)2^k + 2$ for $k \geq 1$. □

We can show that for $k = 1$, $k = 2$, and $k = 3$, we have that $S\left(2^k - 1\right) = G_k$. This supports the following conjecture.

Conjecture 2.5. *We propose that $S\left(2^k - 1\right) = G_k$ for all positive integer k, which implies that Lemma 2.4 achieves equality, that is $P\left(2^k - 1\right) = 2^{(2-k)2^k - 2}$.*

Along with the evidence from small k shown below, we get computational evidence for this conjecture in our results from the linear program of the restricted version of the problem, where all of the elements are unit fractions, in Section 4. By solving the linear program computationally, we verify that $S'\left(2^k - 1\right) = G_k$ for $k \leq 8$.

Proposition 2.6. *For $k = 1$, the multiset $S\left(2^k - 1\right) = G_1$.*

Proof. From Lemma 2.2, $S(1) = \{1\}$. ☐

Proposition 2.7. *For $k = 2$, the multiset $S\left(2^k - 1\right) = G_2$.*

Proof. From Lemma 2.2, we know that the element 1 is included in $S(3)$, so each additional element is at most $\frac{1}{2}$. The multiset $\{1, \frac{1}{2}, \frac{1}{2}\}$ is feasible. So $S(3)$ must equal $\{1, \frac{1}{2}, \frac{1}{2}\}$, which is equivalent to G_2. ☐

Proposition 2.8. *For $k = 3$, the multiset $S\left(2^k - 1\right) = G_3$.*

Proof. We proceed by contradiction. Assume for the sake of contradiction that $S(7) \neq G(3)$. We define intervals where the elements of $S(7)$ could fall. We refer to N_I as the number of elements in $S(7)$ that fall into interval I and s_I as the sum of the elements in $S(7)$ that fall into the same interval.

From Lemma 2.2, we know that the element 1 must be in $S(7)$; it follows that $N_{\left(\frac{1}{2}, 1\right)} = 0$. This leaves us with six elements to place in the interval $\left(0, \frac{1}{2}\right]$.

By Lemma 2.3 the condition

$$N_{\left(\frac{1}{2^i}, 1\right]} > 2^i - 1$$

must be satisfied for some $0 < i < 3$. We have that $N_{\left(\frac{1}{2^1}, 1\right]}$ can contain at most one element, so the condition can't be satisfied for $i = 1$. Hence the condition must be satisfied for $i = 2$: we must have $N_{\left(\frac{1}{4}, 1\right]} > 3$. Since 1 must be included in $S(7)$, this inequality becomes $N_{\left(\frac{1}{4}, \frac{1}{2}\right]} > 2$. From the sum constraint from Definition 1.1, we have that $s_{\left(\frac{1}{4}, \frac{1}{2}\right]} \leq 1$. Then, since every element in $\left(\frac{1}{4}, \frac{1}{2}\right]$ is greater than $\frac{1}{4}$, we can have at most three elements in this interval, so we have $N_{\left(\frac{1}{4}, \frac{1}{2}\right]} = 3$.

From this we can put an upper bound of $\frac{1}{3}$ on the arithmetic mean of elements in $\left(\frac{1}{4}, \frac{1}{2}\right]$. By the Arithmetic Mean-Geometric Mean Inequality (AM-GM), the geometric mean of these three elements is at most $\frac{1}{3}$. There are three elements remaining in $S(7)$ that must all be bounded above by $\frac{1}{4}$, so we can upper bound $P(7)$ by $1\left(\frac{1}{3}\right)^3\left(\frac{1}{4}\right)^3$, which implies that $P(7) < Q(G_3)$, so we have reached a contradiction. ☐

3 Multisets $S(n)$ for General n

In this section, we explore some upper and lower bounds for $P(n)$ for general n in order to examine the asymptotic behavior of $P(n)$. Our lower bound on $P(n)$ is derived from a greedy feasible multiset we can construct for n, and our upper bound on $P(n)$ is derived from upper bounds on each element in $S(n)$. We combine these upper and lower bounds to prove that the negative log of $P(n)$ grows as $n \log n$. Further, we conjecture a tighter upper bound as an extension of Conjecture 2.5. Finally, we compare two feasible multiset candidates for $S(n)$ and show that one always attains a higher product than the other.

Lemma 3.1. *For all positive integers n, $P(n)$ is lower-bounded by $2^{2 \cdot 2^{l(n)} - (n+1)l(n) - 2}$.*

Proof. Given some integer n, we can construct a multiset $L(n)$ greedily with n elements as follows: our first $2^{l(n)} - 1$ elements are the elements of $G_{l(n)}$, and the remaining $n - 2^{l(n)} + 1$ elements are $\frac{1}{2^{l(n)}}$. Since $L(n)$ is feasible, $P(n)$ is lower-bounded by the product of the terms of $L(n)$, which is $2^{2 \cdot 2^{l(n)} - (n+1)l(n) - 2}$. □

Now we would like to upper-bound each element of $S(n)$ by studying the capacity of specific intervals.

Lemma 3.2. *The ith largest element in $S(n)$ is less than $\left(\frac{1}{2}\right)^{l(i)-1}$ for integer $2 \le i \le n$.*

Proof. The sum of the elements of $S(n)$ that fall in the interval $[(\frac{1}{2})^{i+1}, (\frac{1}{2})^{i})$ is constrained to be at most 1. We examine the elements that fit into each of these intervals. An element in the interval $[\left(\frac{1}{2}\right)^{i+1}, \left(\frac{1}{2}\right)^{i})$ can be no smaller than $\left(\frac{1}{2}\right)^{i+1}$, so there can be at most $\frac{1}{\left(\frac{1}{2}\right)^{i+1}}$, or 2^{i+1} elements in this interval.

Then, $\left[\left(\frac{1}{2}\right)^{i+j}, \left(\frac{1}{2}\right)^{i}\right)$ can contain at most $\sum_{k=1}^{j} 2^{i+k}$ elements. Setting $i = 0$, we have that the interval $\left[\left(\frac{1}{2}\right)^{j}, 1\right)$ can contain at most $\sum_{k=1}^{j} 2^{k} = 2^{j+1} - 2$ elements, and the interval $\left[\left(\frac{1}{2}\right)^{j}, 1\right]$ can contain at most $2^{j+1} - 1$ elements. Thus, every element other than the $2^{j+1} - 1$ largest elements must be less than $\left(\frac{1}{2}\right)^{j}$.

We can rewrite the statement above for the ith largest element by setting j to $l(i) - 1$: we have that $i \ge 2^{l(i)}$, then the ith largest element is less than $\left(\frac{1}{2}\right)^{l(i)-1}$. □

Using the above lemma, we may compute a strict upper bound on $P(n)$ by combining the upper bound on every element in $S(n)$.

Lemma 3.3. *For $n \ge 2$, $P(n)$ is strictly upper-bounded by $2^{n - n \cdot l(n) + 2^{l(n)+1} - l(n) - 3}$.*

Proof. We have from Lemma 3.2 that the ith largest element in $S(n)$ is strictly smaller than $2^{1-l(n)}$ for $2 \le i \le n$. Then we have that $P(n) < \Pi_{i=2}^{n} 2^{1-l(i)}$, which means that

$$\lg P(n) < \sum_{i=2}^{n} (1 - l(i)) = n - 1 - \sum_{i=1}^{n} l(i).$$

We have that

$$\begin{aligned}\sum_{i=1}^{n} l(i) &= \left(\sum_{j=1}^{l(n)-1} 2^j j\right) + \left(n - 2^{l(n)} + 1\right) l(n)\\ &= nl(n) - 2^{l(n)+1} + l(n) + 2,\end{aligned}$$

so

$$\lg P(n) < n - nl(n) + 2^{l(n)+1} - l(n) - 3,$$

and therefore $P(n) < 2^{n-nl(n)+2^{l(n)+1}-l(n)-3}$. □

Because the above bounds on $P(n)$ take the form of a shrinking exponential, it is simpler to examine the bounds on $-\lg P(n)$. To formalize asymptotic growth, we recall the formal definitions of Big O, Big Ω, and Big Θ notation.

Definition 3.4. *For functions f and g, when $f(n) = O(g(n))$, the function $f(n)$ is bounded above by $k \cdot g(n)$ for some constant k and large enough n. Similarly, when $f(n) = \Omega(g(n))$, we have that $f(n)$ is bounded below by $k \cdot g(n)$ for some constant k and large enough n. When $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$, then $f(n) = \Theta(g(n))$.*

Theorem 3.5. *The growth of $-\lg P(n)$ is $\Theta(n \lg n)$.*

Proof. Examining the upper bound for $P(n)$ from Lemma 3.3, we see that

$$\begin{aligned}-\lg P(n) &\geq -n + n \cdot l(n) - 2^{l(n)+1} + l(n) + 3\\ &\geq -n + n \cdot l(n) - 2^{l(n)+1}\\ &\geq -n + n \cdot l(n) - 2n\\ &\geq n(l(n) - 3)\\ &\geq n(\lg n - 4)\\ &= \Omega(n \lg n).\end{aligned}$$

Examining the lower bound for $P(n)$ from Lemma 3.1, we see that

$$\begin{aligned}-\lg P(n) &\leq -2 \cdot 2^{l(n)} + (n+1)l(n) + 2\\ &\leq (n+1)l(n) + 2\\ &\leq (n+1)\lg n + 2\\ &= O(n \lg n).\end{aligned}$$

Since $-\lg P(n)$ is both $\Omega(n \lg n)$ and $O(n \lg n)$, we have that $-\lg P(n) = \Theta(n \lg n)$. □

Given that the element-wise upper bound from Lemma 3.3 is created from the bounding of individual elements, we wonder if there exists a tighter upper bound that more fully accounts for the constraints placed on elements in $S(n)$ as a whole. Recall that Conjecture 2.5 predicts that $P\left(2^k - 1\right) = 2^{(2-k)2^k - 2}$. We can rewrite this expression for $P\left(2^k - 1\right)$ in terms of n to get $P(n) = 2^{2n-(n+1)\lg(n+1)}$. This naturally leads to the following conjecture.

Conjecture 3.6. *For $n \geq 1$, $P(n)$ is upper-bounded by $2^{2n-(n+1)\lg(n+1)}$.*

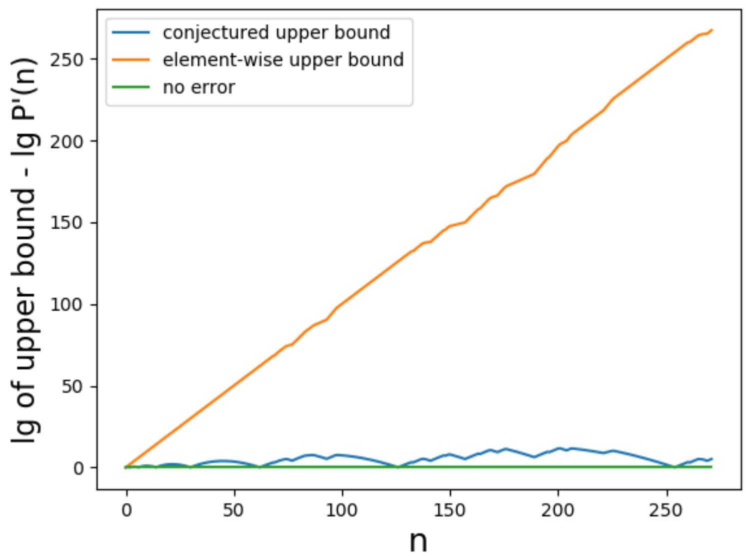

Figure 1: The differences of the logs of the upper bounds to $\lg P'(n)$ are much smaller for the conjectured upper bound than for the element-wise upper bound.

This conjecture is true for the first 273 values of $P'(n)$, as verified computationally using the linear program presented in Section 4. Furthermore, this conjectured bound is much tighter with respect to $P'(n)$ than the element-wise upper bound as Figure 1 demonstrates.

We now explore two potential solutions to $S(n)$, using Conjecture 2.5 as a starting point. One solution is the greedy feasible multiset, while the other is only greedy through the first $2^{k-1} - 1$ elements, where $k = l(n)$, before switching to an alternate form.

Theorem 3.7. *Let $n = 2^k - 1 + c$, where $1 \le c < 2^{k-1}$. Consider two multisets A and B. Let A be constructed as follows: the first $2^k - 1$ elements are the elements of G_k, and the remaining c elements are $\frac{1}{2^k}$. Let B be constructed as follows: the first $2^{k-1} - 1$ elements are the elements of G_{k-1}, and the remaining $2^{k-1} + c$ elements are $\frac{1}{2^{k-1}+c}$. Then $Q(A) > Q(B)$.*

Proof. Because the largest $2^{k-1} - 1$ elements of both multisets A and B are identical, we need to examine only the products of the smallest $2^{k-1} + c$ elements of each multiset in order to show that $Q(A) > Q(B)$. Let the products of the smallest $2^{k-1} + c$ elements of A and B be denoted q_A and q_B, respectively. Thus we wish to show that $q_A > q_B$. We can write q_A as

$$q_A = \left(\frac{1}{2^{k-1}}\right)^{2^{k-1}} \times \left(\frac{1}{2^k}\right)^c = 2^{-[(k-1)\cdot 2^{k-1} + c\cdot k]},$$

and we can write q_B as

$$q_B = \left(\frac{1}{2^{k-1} + c}\right)^{2^{k-1}+c} = \left(2^{k-1} + c\right)^{-\left(2^{k-1}+c\right)}.$$

Define $r = \frac{q_A^{-1}}{q_B^{-1}}$. To show that $q_A > q_B$, we must show that $r < 1$ for all $1 \leq c < 2^{k-1}$.

We have that

$$r = \frac{2^{(k-1)2^{k-1}+kc}}{(2^{k-1}+c)^{2^{k-1}+c}} = \left(\frac{2^{k-1}}{2^{k-1}+c}\right)^{2^{k-1}+c} \cdot 2^c$$

and to simplify this expression, we substitute 2^{k-1} with d to get

$$r = 2^c \left(\frac{d}{d+c}\right)^{d+c}.$$

We can show that $r < 1$ for all $1 \leq c < 2^{k-1}$ if we show that $r < 1$ when $c = 1$ and show that the partial derivative of r with respect to c is always negative for $1 \leq c < 2^{k-1}$.

When $c = 1$, the expression for r becomes $2\left(1 - \frac{1}{d+1}\right)^{d+1}$, which is well-known to be monotonically increasing for $d > 0$ ($d = 2^{k-1} \geq 1$ for our purposes, so the relevant part of the function is always monotonically increasing) and convergent to $\frac{2}{e}$ as k increases. Thus $r < \frac{2}{e} < 1$ when $c = 1$.

Since we've shown that $r < 1$ when $c = 1$, we now want to show that the partial derivative of r with respect to c is negative. We find that the partial is

$$\frac{\partial}{\partial c} 2^c \left(\frac{d}{d+c}\right)^{d+c} = 2^c \left(\frac{d}{d+c}\right)^{d+c} \left(\ln \frac{2d}{c+d} - 1\right).$$

When $1 \leq c < d$ we have that $1 < \frac{2d}{c+d} < 2$ so $\left(\ln \frac{2d}{c+d} - 1\right)$ is always negative. Since $2^c \left(\frac{d}{d+c}\right)^{d+c}$ is always positive for $c, d > 0$, the partial $2^c \left(\frac{d}{d+c}\right)^{d+c} \left(\ln \frac{2d}{c+d} - 1\right)$ is always negative for $1 \leq c < d$. Thus $r < 1$ for all c such that $1 \leq c < d = 2^{k-1}$, and we have shown that $q_A > q_B$ for all c such that $1 \leq c < 2^{k-1}$. Then for all n, we have that $Q(A) > Q(B)$. □

4 Restricting the Problem to Unit Fractions

In this section, we consider the restricted version of this problem described in Definition 1.2 in which we restrict the multiset to only include elements of the form $\frac{1}{c}$, for positive integer c. With this additional constraint, we can easily formulate the problem as a linear program. Furthermore, we note that we only have to consider elements $1, \frac{1}{2}, \frac{1}{3} \ldots, \frac{1}{2n}$ as possible elements in the multiset $S'(n)$ since all smaller terms will never be included as shown in the following lemma.

Lemma 4.1. *For all $x \in S'(n)$, we have that $x \geq \frac{1}{2n}$.*

Proof. We prove this lemma by contradiction. Assume for the purposes of contradiction that there exists an element $x \in S'(n)$ such that $x < \frac{1}{2n}$. We construct a multiset $F(n)$ that is identical to $S'(n)$, except that the element x is replaced with $x' = \frac{1}{2n}$. The multiset $F(n)$ is feasible unless element x' and some of the other elements in the interval $\left[\frac{1}{4n}, \frac{1}{n}\right)$ sum to more than 1. However, any n terms in the above interval must sum to less than 1. Thus, $F(n)$ is feasible, and $Q(F(n)) > P'(n)$, so we have reached a contradiction. □

For $j = 1, \ldots, 2n$ let variable p_j indicate the number of times the fraction $\frac{1}{j}$ appears in $S'(n)$.

We have that

$$P'(n) = \prod_{j=1}^{2n} \left(\frac{1}{j}\right)^{p_j}.$$

Below, we present the linear program that we use to compute $P'(n)$. First, we rewrite our objective so that it is linear.

$$-\lg(P'(n)) = -\lg\left(\prod_{j=1}^{2n} \left(\frac{1}{j}\right)^{p_j}\right) = \sum_{j=1}^{2n} p_j \lg j.$$

We construct a primal linear program which defines the optimization problem whose solution is $S'(n)$:

$$\begin{array}{lll}
\min & \displaystyle\sum_{j=1}^{2n} p_j \lg j & \\
\text{subject to} & \displaystyle\sum_{j=i}^{\max(2n,2i-1)} p_j \times \frac{1}{j} \leq 1 & i = 1, \ldots, 2n \\
& \displaystyle\sum_{j=1}^{2n} p_j = n & \\
& p_j \geq 0 & j = 1, \ldots, 2n.
\end{array}$$

In order to bound the value of the objective of the primal, we construct the dual of the above primal linear program:

$$\begin{array}{lll}
\max & \displaystyle\left(\sum_{i=1}^{2n} d_i\right) + n \times d^{(t)} & \\
\text{subject to} & \displaystyle\left(\sum_{i=\lceil\frac{j+1}{2}\rceil}^{j} d_i \times \frac{1}{j}\right) + d^{(t)} \leq \lg j & j = 1, \ldots, 2n \\
& d_i \leq 0 & i = 1, \ldots, 2n \\
& d^{(t)} \text{ unrestricted in sign.} &
\end{array}$$

We notice that the direction of inequality of the first constraint of the dual is the same as that of the primal. This is because the variables of the dual are restricted to be nonpositive, rather than nonnegative as in standard form.

Since the primal is feasible and bounded, we know that the dual must be feasible. By Weak Duality, any value that the objective of the dual can achieve is a lower bound on the objective of the primal, which produces an upper bound on $P'(n)$.

Lemma 4.2. *The following is a feasible solution for the dual linear program for $n = 2^k - 1$ for some integer k. We define the function $f(x) = \left(\lg x - d^{(t)}\right) x$. Our solution is as follows:*

$$
\begin{aligned}
&d^{(t)} = k - 1,\\
&d_{2^{k-2}} = f\left(2^{k-2}\right),\\
&d_{2^{k-3}} = f\left(2^{k-1-\lg e}\right),\\
&d_{2^i} = f\left(2^{i+1} - 1\right) \text{ for } 0 \le i \le k-4,\\
&\text{and all other variables of the dual are 0.}
\end{aligned}
$$

Proof. First, we can easily check that all variables other than $d^{(t)}$ are nonpositive.

Now, we check if the proposed solution to the dual satisfies the first dual constraint. Let j be any integer in $\{1, 2, \ldots, 2^{k-1}-1\}$. Let z be the integer in $\{0, 1, 2, \ldots k-2\}$ such that $j \in I_z = [2^z, 2^{z+1}-1]$.

We will demonstrate that the first dual constraint

$$
\left(\sum_{i=\lceil \frac{j+1}{2} \rceil}^{j} d_i \times \frac{1}{j}\right) + d^{(t)} \le \lg j \implies f(j) \ge \sum_{i=\lceil \frac{j+1}{2} \rceil}^{j} d_i
$$

is satisfied for all $j \in \{1, 2, \ldots, 2^{k-1} - 1\}$. Note that doing so suffices to finish the proof of the lemma, since the condition is always satisfied for $j > 2^{k-1}-1$ because $\lg j \ge d^{(t)} = k-1$ for all such j.

The function $f(x)$ decreases when $x \in \left(0, 2^{d^{(t)}-\lg e}\right)$, reaches a minimum at $x = 2^{d^{(t)}-\lg e} \approx 2^{k-2.44}$, and increases from that point onward. We split into cases based on the behavior of the function f at different values of z.

Case 1: $z = k-2$. Then, $j \in [2^{k-2}, 2^{k-1}-1]$. The first dual constraint becomes $f(j) \ge \sum_{i=\lceil \frac{j+1}{2} \rceil}^{j} d_i$. The only term d_i in this sum that is nonzero is $d_{2^{k-2}}$. Thus, in this case the first dual constraint becomes $f(j) \ge d_{2^{k-2}} = f(2^{k-2})$. Since $f(x)$ is increasing on the interval $\left[2^{k-2}, 2^{k-1} - 1\right]$, the constraint is always satisfied in this case.

Case 2: $z = k-3$. Then $j \in \left[2^{k-3}, 2^{k-2} - 1\right]$. The constraint becomes $f(j) \ge f\left(2^{k-1-\lg e}\right)$. Since $f\left(2^{k-1-\lg e}\right)$ is the global minimum of $f(x)$, the constraint is always satisfied in this case.

Case 3: $z \le k-4$. Then $j \in \left[2^z, 2^{z+1} - 1\right]$. The constraint becomes $f(j) \ge f\left(2^{z+1} - 1\right)$. Since $f(x)$ is decreasing on the interval $\left(0, 2^{k-3} - 1\right]$, the constraint is always satisfied in this case. □

Now that we have found a feasible solution to the dual, we use the objective value that the dual solution achieves to lower bound the optimal value of the objective of the primal linear program.

Theorem 4.3. *For integer $k \ge 2$ and $n = 2^k - 1$, we have that $P'(n)$ is upper bounded by*

$$
\frac{2^{2^{k-2}} \left(\frac{2^{k-1}}{2^{k-1-\lg e}}\right)^{2^{k-1-\lg e}} \prod_{i=1}^{k-3} \left(\frac{2^{k-1}}{2^i-1}\right)^{2^i-1}}{2^{(2^k-1)(k-1)}}.
$$

Proof. For the dual solution given in Lemma 4.2, the objective value simplifies as follows:

$$J = \left(\sum_{i=1}^{2n} d_i\right) + n \times d^{(t)} = \sum_{i=1}^{k-3} \left[\left(\lg(2^i - 1) - k + 1\right)\left(2^i - 1\right)\right] + \left(\lg\left(2^{k-2}\right) - k + 1\right)\left(2^{k-2}\right) + \left(\lg\left(2^{k-1-\lg e}\right) - k + 1\right)\left(2^{k-1-\lg e}\right) + \left(2^k - 1\right)(k-1)$$

$$= \sum_{i=1}^{k-3}\left[\lg\left(\frac{2^i - 1}{2^{k-1}}\right)^{2^i - 1}\right] - 2^{k-2} + \lg\left(\frac{2^{k-1-\lg e}}{2^{k-1}}\right)^{2^{k-1-\lg e}} + \left(2^k - 1\right)(k-1)$$

$$= \lg\left(2^{\left(-2^{k-2}\right)}\left(\frac{2^{k-1-\lg e}}{2^{k-1}}\right)^{2^{k-1-\lg e}} \prod_{i=1}^{k-3}\left(\frac{2^i - 1}{2^{k-1}}\right)^{2^i - 1}\right) + \left(2^k - 1\right)(k-1).$$

By Weak Duality, J is a lower bound on the value of the objective function of the primal linear program. The objective of the primal linear program is the negative log of the product $P'(n)$. Thus, 2^{-J}, which is equivalent to the expression given in the statement of the theorem, is an upper bound on $P'(n)$. □

We note that our previous upper bound on $P(n)$ from Lemma 3.3 is also an upper bound on $P'(n)$, since all feasible multisets for the modified problem are also feasible for the original problem. Then, we can compare the upper bounds from Lemma 3.3 and Theorem 4.3 to the computed values of $P'(n)$ for various values of n.

n	k	$P'(n)$ from Solving LP	Lemma 3.3 Upper Bound	Thm. 4.3 Upper Bound
3	2	$P'(3) = \frac{1}{4}$	$P'(3) \leq 1$	$P'(3) \leq 0.52$
7	3	$P'(7) = 9.77 \times 10^{-4}$	$P'(7) \leq 6.25 \times 10^{-2}$	$P'(7) \leq 1.06 \times 10^{-3}$
15	4	$P'(15) = 5.82 \times 10^{-11}$	$P'(15) \leq 9.54 \times 10^{-7}$	$P'(15) \leq 6.90 \times 10^{-11}$
255	8	$\lg P'(255) \approx -1538$	$\lg P'(255) \leq -1284$	$\lg P'(255) \leq -1460.66$

Table 1: We see that the upper bound from Theorem 4.3 is much tighter than the upper bound from Lemma 3.3. In fact, as n increases, the upper bound given by the theorem becomes orders of magnitude smaller than the upper bound given by the lemma.

To visualize the relationship between the upper and lower bounds that we proved for $P(n)$ in Section 3 and the upper bound that we proved for $P'\left(2^k - 1\right)$ in Theorem 4.3, we graph all of the bounds along with the value of $P'(n)$ in Figure 2.

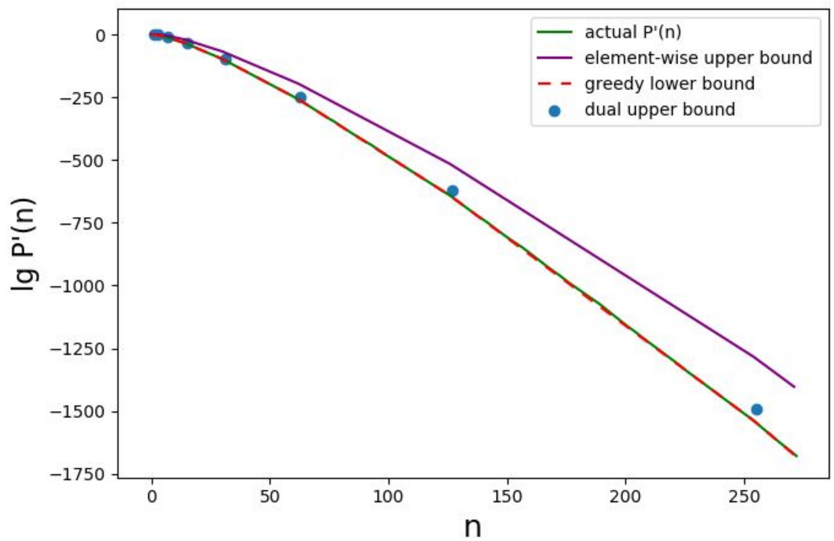

Figure 2: This visualization shows the bounds derived throughout this paper, including the element-wise upper bound on $P(n)$ (from Lemma 3.2), the greedy lower bound on $P(n)$ (from Lemma 3.1), the upper bound on $P'(n)$ (from Theorem 4.3), and the value of $P'(n)$ computed from the primal linear program.

We note that the greedy lower bound and actual $P'(n)$ somewhat overlap on the graph. We show the difference between these two curves in Figure 3.

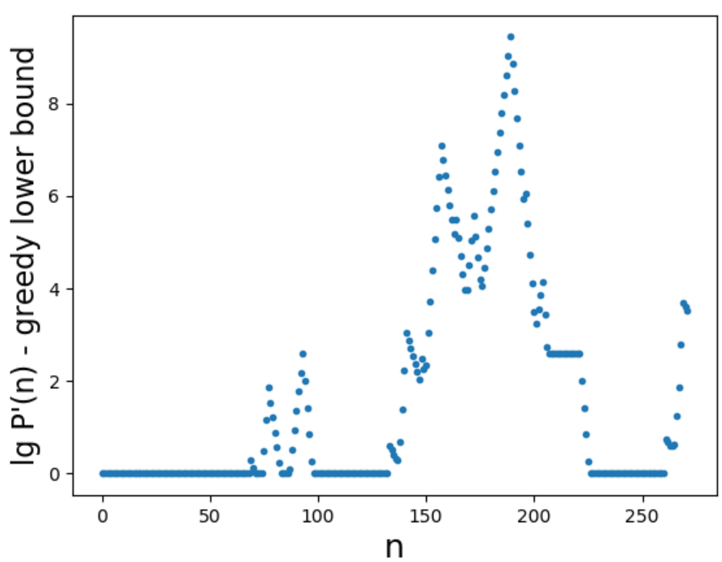

Figure 3: From this visualization, we see that the greedy lower bound becomes a tight lower bound on $P'(n)$ at and around powers of 2.

The graphs show that $P'(n)$ fall between the upper and lower bounds that we proved for $P(n)$. Additionally, the values of $S(1), S(3)$, and $S(7)$ that we proved in Section 2 provide further evidence that the products $P'(n)$ and $P(n)$ may be the same for all positive integer n. This leads us to the following conjecture.

Conjecture 4.4. *For all positive integers n, the multiset $S(n)$ only consists of unit fractions.*

5 Relaxing the Problem to Non-Integer Copies

We seek to relax the constraint that elements must appear in integral numbers of copies from our original problem due to the difficulties that arise when dealing with arbitrarily small modifications to discrete quantities. By allowing for a non-integral number of copies, we have increased fluidity in varying the value and multiplicity of elements by arbitrarily small amounts. This fluidity makes it easier for us to prove that some multisets are suboptimal because we can vary them by sufficiently small degrees to increase their products. Then, we can prove the following theorem for the relaxed problem described in Definition 1.3:

Theorem 5.1. *For any positive integer n and all α such that $\frac{\min S''(n)}{2} < \alpha \leq \max S''(n)$, the interval $I_\alpha = [\alpha, 2\alpha)$ contains at least one element.*

Proof. We proceed by contradiction. Assume for the sake of contradiction that $S''(n)$ contains some interval $[\alpha, 2\alpha)$ with no elements. Let the smallest element greater than or equal to α be $\frac{1}{x}$, and let the largest element less than α be $\frac{1}{y}$. Let $S''(n)$ contain c copies of $\frac{1}{x}$ and d copies of $\frac{1}{y}$. We choose some $0 < \varepsilon < \frac{c \cdot d(y-2x)}{c \cdot y + 2d \cdot x}$, and we construct a new multiset $F''(n)$ that is the same as $S''(n)$, except we replace our c copies of $\frac{1}{x}$ with $c + \varepsilon$ copies of $\frac{1}{x\left(1+\frac{\varepsilon}{c}\right)}$, and we replace our d copies of $\frac{1}{y}$ with $d - \varepsilon$ copies of $\frac{1}{y\left(1-\frac{\varepsilon}{d}\right)}$.

Figure 4: We show where $\frac{1}{x}$ and $\frac{1}{y}$, as well as their modified replacements $\frac{1}{x\left(1+\frac{\varepsilon}{c}\right)}$ and $\frac{1}{y\left(1-\frac{\varepsilon}{d}\right)}$, fall on a number line.

To reach a contradiction, it suffices to prove the following two claims: **Claim 1:** $F''(n)$ is feasible and **Claim 2:** $Q(F''(n)) > P''(n)$.

Proof of Claim 1. When $\varepsilon < \frac{c \cdot d(y-2x)}{c \cdot y + 2d \cdot x}$, we have $\frac{1}{x\left(1+\frac{\varepsilon}{c}\right)} > 2 \times \frac{1}{y\left(1-\frac{\varepsilon}{d}\right)}$. Thus, the new values that we introduced will not be in the same interval $[\alpha', 2\alpha')$ with each other for any α'.

Further, new conflicts with values in $F''(n)$ outside of the interval $\left[\frac{1}{y}, \frac{1}{x}\right]$ will not be introduced since the sums of copies of the two replaced values remain $\frac{c}{x}$ and $\frac{d}{y}$, and any element that these new values interact with must have also been in the same interval $[\alpha', 2\alpha')$ as the original values.

Proof of Claim 2. We only consider the product of the terms which are different between $S''(n)$ and $F''(n)$. Hence, we must show $\left(\frac{1}{x}\right)^c \left(\frac{1}{y}\right)^d < \left(\frac{1}{x\left(1+\frac{\varepsilon}{c}\right)}\right)^{c+\varepsilon} \left(\frac{1}{y\left(1-\frac{\varepsilon}{d}\right)}\right)^{d-\varepsilon}$. We have that

$$\varepsilon < \frac{cd(y-2x)}{cy+2dx} < \frac{cd(y-x)}{cy+dx}$$

which is equivalent to

$$\frac{1-\frac{\varepsilon}{d}}{1+\frac{\varepsilon}{c}} > \frac{x}{y}.$$

Because it is well-known that $\left(1+\frac{1}{n}\right)^n$ and $\left(1-\frac{1}{n}\right)^n$ are monotonically increasing and convergent to e and $\frac{1}{e}$ for all $n \geq 1$, respectively, we know that $\left(1+\frac{\varepsilon}{c}\right)^{\frac{c}{\varepsilon}} < e$ and $\left(1-\frac{\varepsilon}{d}\right)^{\frac{d}{\varepsilon}} < \frac{1}{e}$ for all $\varepsilon \leq d$. To verify that our chosen ε is always less than or equal to d, we can start with the fact that

$$x(d+c) > 0,$$

so

$$cy - cx < cy + dx,$$

and we rearrange to get that

$$\frac{cd(y-x)}{cy+dx} < d.$$

Then, since $\varepsilon < \frac{cd(y-x)}{cy+dx}$, we have that $\varepsilon \leq d$. Now we can multiply the inequalities $\left(1+\frac{\varepsilon}{c}\right)^{\frac{c}{\varepsilon}} < e$ and $\left(1-\frac{\varepsilon}{d}\right)^{\frac{d}{\varepsilon}} < \frac{1}{e}$ together with $\frac{x}{y} < \frac{1-\frac{\varepsilon}{d}}{1+\frac{\varepsilon}{c}}$ from above to get that

$$\frac{1-\frac{\varepsilon}{d}}{1+\frac{\varepsilon}{c}} > \frac{x}{y}\left(1+\frac{\varepsilon}{c}\right)^{\frac{c}{\varepsilon}}\left(1-\frac{\varepsilon}{d}\right)^{\frac{d}{\varepsilon}},$$

which, after we take both sides to the ε power, transforms into

$$\left(\frac{1}{x}\right)^c \left(\frac{1}{y}\right)^d < \left(\frac{1}{x\left(1+\frac{\varepsilon}{c}\right)}\right)^{c+\varepsilon} \left(\frac{1}{y\left(1-\frac{\varepsilon}{d}\right)}\right)^{d-\varepsilon}$$

which finishes the proof of the second claim. □

If we could prove the strengthened theorem that for any positive integer n and all α such that $\frac{\min S''(n)}{2} < \alpha \leq \max S''(n)$, the interval $I_\alpha = [\alpha, 2\alpha)$ contains exactly one distinct element, then the following conjecture holds.

Conjecture 5.2. *All values in the multiset $S''(n)$ take the form $x, \frac{x}{2}, \frac{x}{4}, \frac{x}{8}, \ldots$*

Note that if Conjecture 5.2 holds, then a multiset is optimal if and only if we insert elements greedily, starting with some number x: for example, $\frac{1}{x}$ copies of x, $\frac{2}{x}$ copies of $\frac{x}{2}$, and so on.